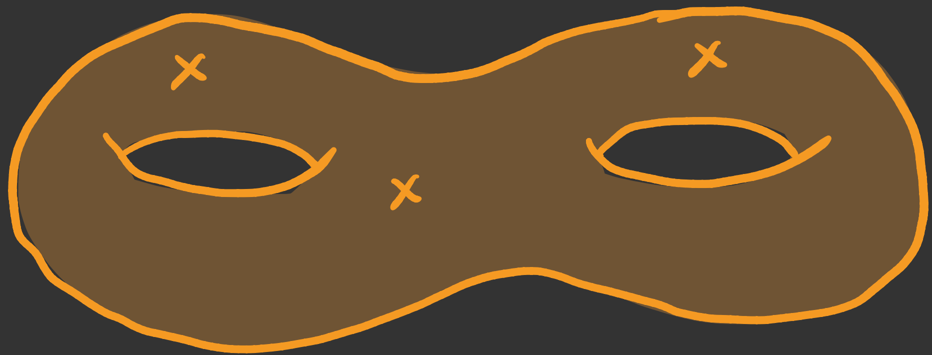
Modular curves



Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup.

The corresponding modular curve $\mathbf{Y(\Gamma)}$ is defined as the quotient space

$$Y(\Gamma) := \Gamma \backslash \mathfrak{h}$$

Today we will show that $Y(\Gamma)$ can be

- made into a **Riemann surface**
- compactified

The resulting compact Riemann surface is denoted $\mathbf{X(\Gamma)}$

$\mathfrak{h}$ has the topology as a subset of $\mathbb{R}^2$.

Let Γ be a congruence subgroup.

$Y(\Gamma) = \Gamma \backslash \mathfrak{h}$ has the topology of a quotient space

Lemma: $Y(\Gamma)$ is a Hausdorff space.

Lemma: Let $z_1, z_2 \in \mathfrak{h}$. Then there exist open sets $\mathcal{U}_1, \mathcal{U}_2$ s.t $z_1 \in \mathcal{U}_1$, $z_2 \in \mathcal{U}_2$ and for all $\gamma \in \Gamma$: if $\mathcal{U}_1 \cap \gamma \mathcal{U}_2 \neq \emptyset \Rightarrow z_1 = \gamma z_2$

Claim: Let $z_1, z_2 \in \mathfrak{h}$ and let U_1' be a nbh of z_1 and U_2' be a nbh of z_2 such that the closure of U_1' and the closure of U_2' are compact. Then the set

$$\{\gamma \in \Gamma \mid U_1' \cap \gamma U_2' \neq \emptyset\} \text{ is finite.}$$

Proof: Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to

Then there exist $\tau_1 \in U_1'$ and $\tau_2 \in U_2'$ such that

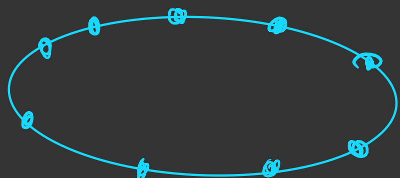
$$\tau_1 = \frac{a\tau_2 + b}{c\tau_2 + d} \Rightarrow$$

$$\Rightarrow \quad \mathrm{Im}(\tau_1) = \mathrm{Im}\left(\frac{a\tau_2 + b}{c\tau_2 + d}\right)$$

$$\mathrm{Im}(\tau_1) = \frac{\mathrm{Im}(\tau_2)}{|c\tau_2 + d|^2}$$

$$|c\tau_2 + d|^2 = \frac{\mathrm{Im}(\tau_2)}{\mathrm{Im}(\tau_1)} \longleftarrow$$ only finitely many $(c,d) \in \mathbb{Z}^2$

satisfy this condition for some $\tau_1 \in \mathcal{U}_1'$, $\tau_2 \in \mathcal{U}_2'$.



$\leftarrow$ ellips depends on τ_1 and τ_2 ($\tau_1 \in \mathcal{U}_1'$ and $\tau_2 \in \mathcal{U}_2'$).

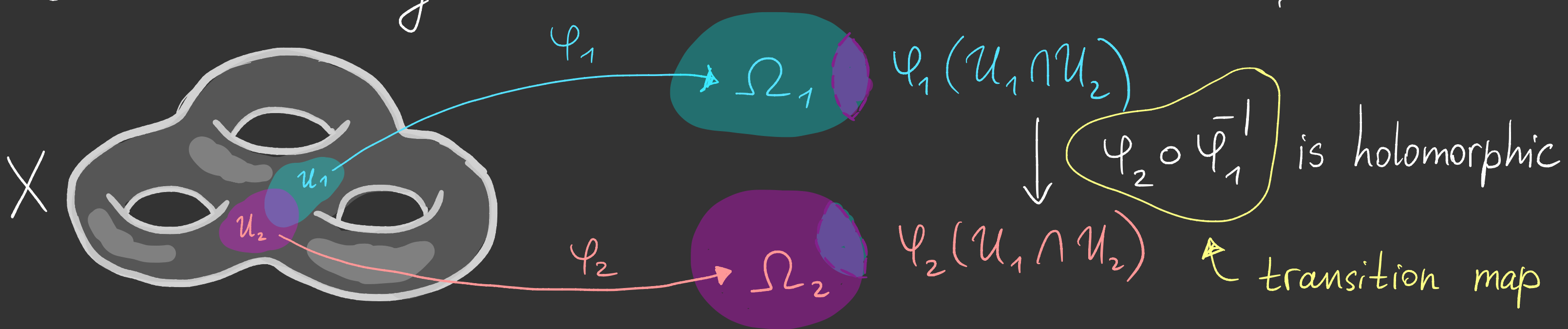
$$\tau_1 = \frac{a\tau_2 + b}{c\tau_2 + d} \Rightarrow -\frac{1}{\tau_1} = \frac{-c\tau_2 - d}{a\tau_2 + b} \Rightarrow \mathrm{Im}\left(-\frac{1}{\tau_1}\right) = \mathrm{Im}\left(\frac{-c\tau_2 - d}{a\tau_2 + b}\right)$$

Only finitely many (a,b) satisfy this condition.

$\blacksquare$ of claim

Definition: A **Riemann surface** X is a complex manifold of dimension 1.

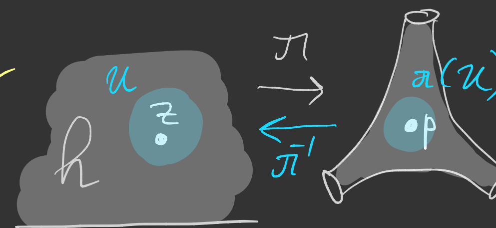
- Second countable topological space (has countable base of topology)
- Hausdorff
- Any point $P \in X$ has a neighborhood $\mathcal{U}_p$ and a homeomorphism $\varphi : \mathcal{U}_p \to \Omega_p$ where Ω_p is an open subset in $\mathbb{C}$ — chart
- Consistency conditions between charts:



Example: Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup. Suppose that for all $z \in \mathfrak{h}$ the stabilizer $\mathrm{Stab}_\Gamma(z)$ is trivial.
Then the modular curve $Y(\Gamma) = \Gamma \backslash \mathfrak{h}$ is a Riemann surface.

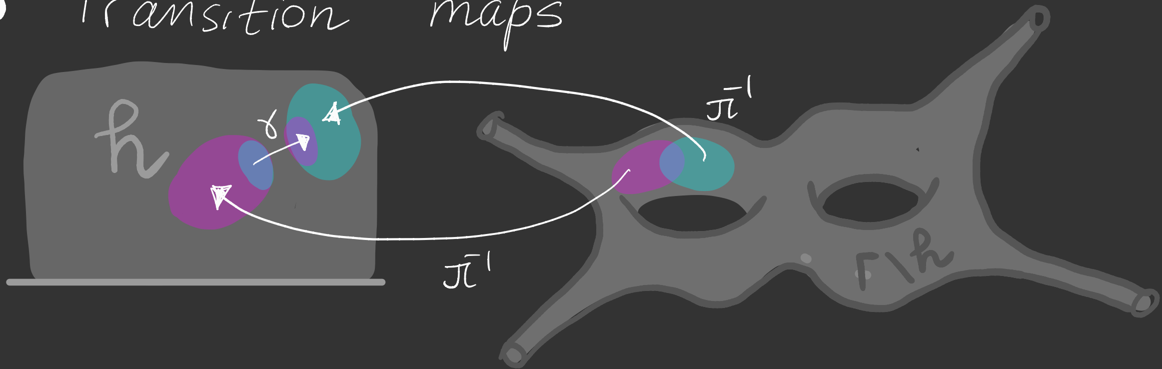
Proof:

- Second countable ✓
- Hausdorff ✓
- Charts



Let $\pi : \mathfrak{h} \to Y(\Gamma)$ be the canonical projection
Let $p \in Y(\Gamma)$ and $z \in \mathfrak{h}$ s.t $\pi(z) = p$
Let $\mathcal{U}$ be a nbh of z s.t for all $\gamma \in \Gamma$:
$\mathcal{U} \cap \gamma\,\mathcal{U} \neq \emptyset$ implies $\gamma = id$.
Then $\pi^{-1} : \pi(\mathcal{U}) \to \mathcal{U}$ is a homeomorphism

- Transition maps



are of the form $z \mapsto \gamma z$ for $\gamma \in \Gamma$.

Möbius transformations $z \mapsto \dfrac{az+b}{cz+d}$ are holomorphic. $\blacksquare$

Definition: A point $z \in \mathfrak{h}$ is an **elliptic point** with respect to a discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ if $Stab_{\Gamma}(z)$ is non-trivial (as a group of transformations). The point $\pi(z) \in Y(\Gamma)$ is also called elliptic.

Lemma: For each elliptic point $z \in \mathfrak{h}$ the group $Stab_{\Gamma}(z)$ is finite cyclic.

Proof: First, we show that the group $Stab_{\Gamma}(z)$ is finite.

$$\frac{az+b}{cz+d} = z \qquad cz^2 + (d-a)z - b = 0$$

$$\Downarrow$$

$$\begin{cases} c \cdot \mathrm{Re}(z^2) + (d-a)\,\mathrm{Re}(z) - b = 0 \\ c \cdot \mathrm{Im}(z^2) + (d-a)\,\mathrm{Im}(z) = 0 \end{cases}$$

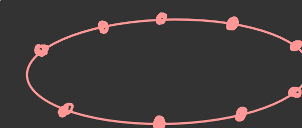
$$z^2 = \left(\mathrm{Re}\,z + i\,\mathrm{Im}(z)\right)^2 = \mathrm{Re}^2(z) + 2i\,\mathrm{Re}(z)\cdot\mathrm{Im}(z) - \mathrm{Im}^2(z)$$

$$2\,\mathrm{Re}(z)\,\mathrm{Im}(z)\,c + (d-a)\,\mathrm{Im}(z) = 0 \Rightarrow d-a = -2\,\mathrm{Re}(z)\cdot c$$

$$c\left(\mathrm{Re}^2(z) - \mathrm{Im}^2(z)\right) - 2\,\mathrm{Re}^2(z)\cdot c - b = 0$$

$$\Downarrow$$

$$b = -c\left(\mathrm{Re}^2(z) + \mathrm{Im}^2(z)\right) = -c\cdot|z|$$



$$\det\begin{pmatrix} d + 2\,\mathrm{Re}(z)c & -c|z| \\ c & d \end{pmatrix} = 1 \qquad \underbrace{d^2 + 2\,\mathrm{Re}(z)\,cd + c^2|z| = 1}$$

Only finitely many $(c,d)\in\mathbb{Z}^2$ can satisfy this condition.

Now, we prove that $\mathrm{Stab}_\Gamma(z)$ is cyclic.

Let $\gamma \in \mathrm{Stab}_\Gamma(z)$. We write

because z is a fixed point of γ

our ellipt pt

local coordinate around z in $\mathfrak{h}$

$$\gamma(z+w) = \sum_{n=0}^{\infty} a_n(\gamma)\, w^n = z + \sum_{n=1}^{\infty} a_n(\gamma)\, w^n$$

Note that $\gamma : \mathfrak{h} \to \mathfrak{h}$ is a bijection $\Rightarrow a_1(\gamma) \neq 0$.

Consider the function $a_1 : \mathrm{Stab}_\Gamma(z) \longrightarrow \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$

group w.r.t mult.

We observe that this function is a homomorphism.

Indeed,

$$\gamma_1 \circ \gamma_2 (z+w) = \sum_{n=0}^{\infty} a_n(\gamma_1)\big(\gamma_2(z+w) - z\big)^n$$

$$= z + \sum_{n=1}^{\infty} a_n(\gamma_1)\big(a_1(\gamma_2)\, w + \overline{O}(w)\big)^n$$

$$= z + a_1(\gamma_1)\cdot a_1(\gamma_2)\, w + \overline{O}(w)$$

Next, we show that $a_1 : \mathrm{Stab}_\Gamma(z) \to \mathbb{C}^\times$ is injective.

⚠ Suppose that $a_1(\gamma) = 1$ for some $\gamma \in \mathrm{Stab}_\Gamma(z) \setminus \{id\}$.

Since the group $\mathrm{Stab}_\Gamma(z)$ is finite, there exists an integer number $n \geq 1$ such that $\gamma^n = id$.

Suppose that

$$\gamma(z+w) = z + w + a_m w^m + o(w^m)$$

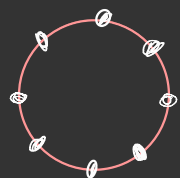
the first nonzero coefficient with $m > 1$

$$\begin{aligned}
\gamma \circ \gamma(z+w) - z &= (\gamma(z+w) - z) + a_m (\gamma(z+w) - z)^m + o(w^m) \\
&= (w + a_m w^m + o(w^m)) + a_m (w + a_m w^m + (w^m))^m + o(w^m) \\
&= w + 2 a_m w^m + o(w^m)
\end{aligned}$$

$$\underbrace{\gamma \circ \gamma \cdots \circ \gamma}_{n \text{ times}}(z+w) = z + w + n \cdot a_m w^m + o(w^m)$$

Now $\gamma^n = id$ implies $a_m = 0$. ↯ with ⚠. Therefore a_1 is injective.

Since $a_1 : \mathrm{Stab}_\Gamma(z) \to \mathbb{C}^\times$ is injective and $\mathbb{C}^\times$ contains only cyclic finite subgroups,



$\mathbb{C}_1 := \{ z \mid |z| = 1 \}$

We conclude that $\mathrm{Stab}_\Gamma(z)$ is a finite cyclic group $\blacksquare$

Claim: Each point $z \in \mathfrak{h}$ has a neighbourhood $\mathcal{U}$ such that for all $\gamma \in \Gamma$:

$\mathcal{U} \cap \gamma\,\mathcal{U} \neq \emptyset$ implies $\gamma \in \mathrm{Stab}_\Gamma(z)$.

Constucting holomorphic charts around an elliptic point z

$$\mathcal{U} = \{ w \mid \rho_{hyp}(z, w) < \varepsilon \}$$

$$\xrightarrow{\quad \varphi \quad}$$

linear fractional transformation

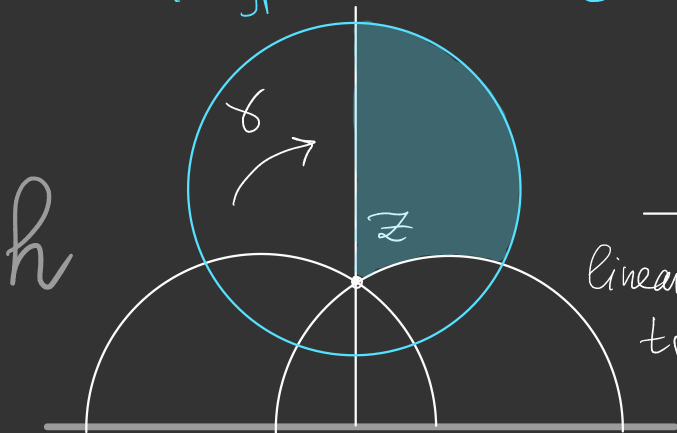
$$\mathcal{D} = \{ w \mid |w| < 1 \}$$

$$\varphi \circ \gamma \circ \varphi^{-1} : \mathcal{D} \to \mathcal{D}$$

$$\varphi \circ \gamma \circ \varphi^{-1}(w) = e^{\frac{2\pi i}{n}} \cdot w$$

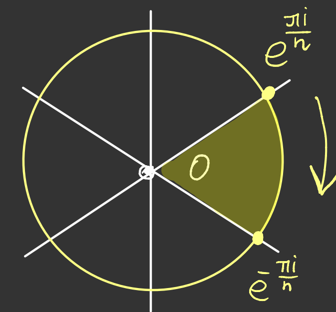
$\mathcal{U} = \{ w \mid \rho_{hyp}(z, w) < \varepsilon \}$

$\mathbb{D} = \{ w \mid |w| < 1 \}$



φ

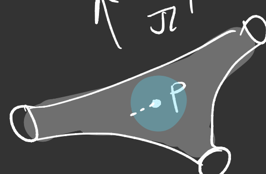
linear fractional transformation



$\tilde{\gamma}(w) = e^{\frac{2\pi i}{n}} \cdot w$

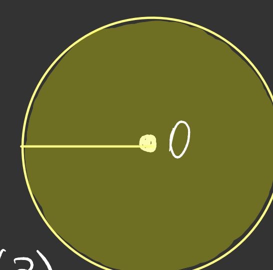
$\gamma^n = id$

π^{-1}



$Y(\Gamma)$

Ψ $\quad z \longmapsto z^n$



$\Psi : \mathbb{D} / \langle e^{\frac{2\pi i}{n}} \rangle \to \mathbb{D}$

is a homeomorphism

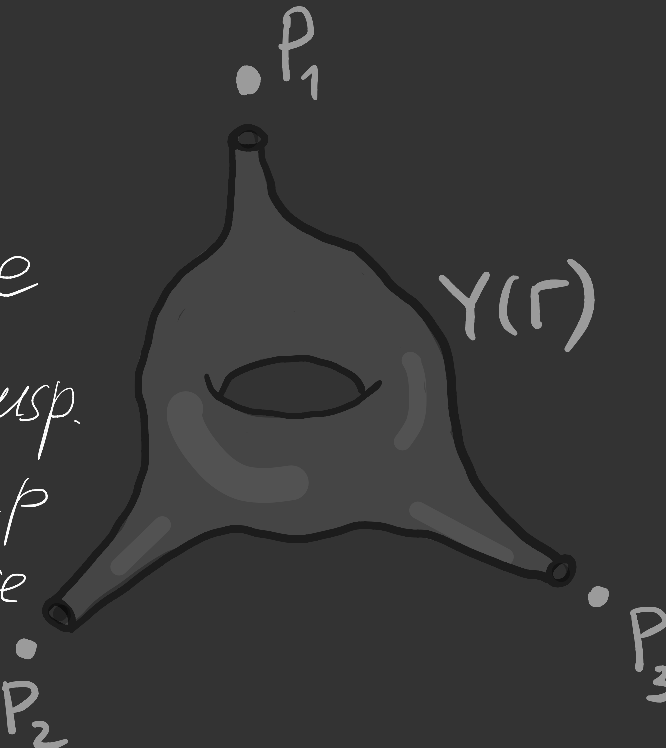
We set $\varphi \circ \psi \circ \pi^{-1} : \pi(\mathcal{U}) \longrightarrow \mathbb{D}$ to be a holomorphic chart about $p = \pi(z)$

Cusps

$$\mathfrak{h}^* := \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$$

$$X(\Gamma) := \Gamma \backslash \mathfrak{h}^*$$

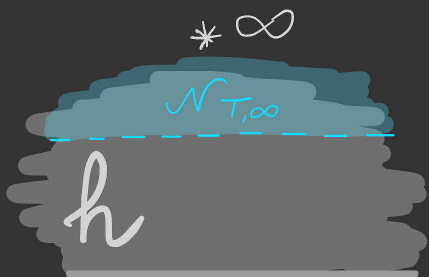
Lemma: The modular curve $X(1) := SL_2(\mathbb{Z}) \backslash \mathfrak{h}^*$ has 1 cusp. For any congruence subgroup Γ of $SL_2(\mathbb{Z})$ the modular curve $X(\Gamma)$ has finitely many cusps.



Neighbourhood of a cusp

First, we consider the cusp ∞

We define the nbh. of ∞ in $\mathfrak{h}^*$:

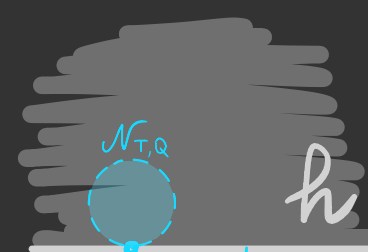


$$\mathcal{N}_{T,\infty} := \{ z \mid \mathrm{Im}(z) > T \}, \ T>0$$

$$\mathcal{N}^*_{T,\infty} := \mathcal{N}_{T,\infty} \cup \infty \quad \text{open in } \mathfrak{h}^*$$

Let c and d be coprime integers and $Q = -\frac{d}{c}$

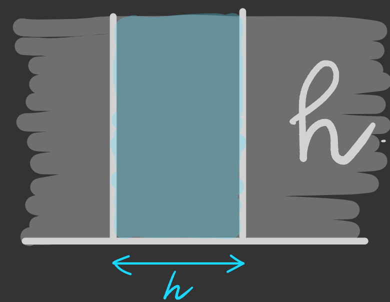
$$ad - bc = 1 \qquad \frac{aQ+b}{cQ+d} = \frac{a(-\frac{d}{c})+b}{c(-\frac{d}{c})+d} = \infty$$



$$\mathcal{N}_{T,Q} := \{ z \mid \mathrm{Im}\left(\frac{az+b}{cz+d}\right) > T \} \qquad \mathcal{N}^*_{T,Q} := \mathcal{N}_{T,Q} \cup Q \qquad Q = -\frac{d}{c}, \ d,c \in \mathbb{Z}$$

$$\mathrm{Stab}_\infty(\Gamma_1) = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$$

$$\mathrm{Stab}_\infty(\Gamma) = \left\{ \begin{pmatrix} 1 & n \cdot h \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\} \text{ for some } h \in \mathbb{Z}_{\geq 1}$$

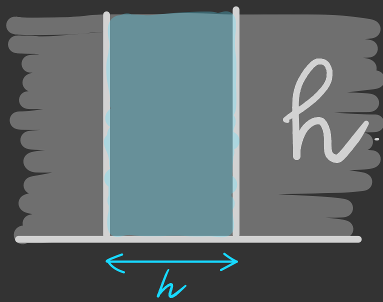


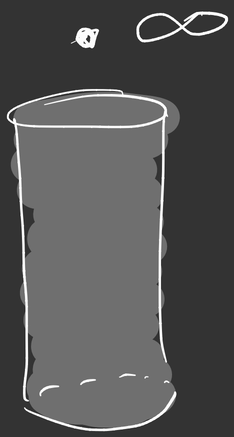
$$X(\Gamma) = \Gamma \backslash \mathfrak{h}^*$$

Lemma: The modular curve $X(\Gamma)$ is Hausdorff, connected and compact.

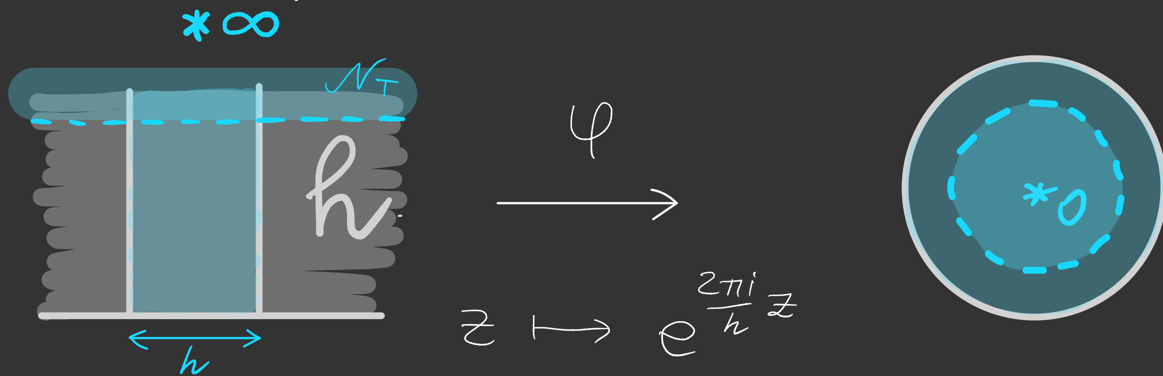
$$\mathrm{Stab}_\infty(\overline{\Gamma}_1) = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \; n \in \mathbb{Z} \right\}$$

$$\mathrm{Stab}_\infty(\overline{\Gamma}) = \left\{ \begin{pmatrix} 1 & n \cdot h \\ 0 & 1 \end{pmatrix} \middle| \; n \in \mathbb{Z} \right\} \text{ for some } h \in \mathbb{Z}_{\geq 1}$$





Holomorphic charts around cusps



$\mathcal{D} = \{ w \mid |w| < 1 \}$

We choose $T > 0$ so that for all $\gamma \in \Gamma$:

$$\mathcal{N}_T \cap \gamma \mathcal{N}_T \neq \emptyset \implies \gamma \in \mathrm{Stab}_\Gamma(\infty)$$

Exercise:

Why can we choose such T?

Lemma: The transition maps between the charts we have defined around

- regular points
- elliptic points
- cusps

are holomorphic